Problems and Solutions

Problems and Solutions. The aim of this section is to encourage readers to participate in the intriguing process of problem solving in Mathematics. This section publishes problems and solutions proposed by readers and editors.

Readers are welcome to submit solutions to the following problems. Your solutions, if chosen, will be published in the next issue, bearing your full name and address. A publishable solution must be correct and complete, and presented in a well-organised manner. Moreover, elegant, clear and concise solutions are preferred.

Readers are also invited to propose problems for future issues. Problems should be submitted with solutions, if any. Relevant references should be stated. Indicate with an ° if the problem is original and with an * if its solution is not available.

All problems and solutions should be typewritten double-spaced, and two copies should be sent to the Editor, Mathematical Medley, c/o Department of Mathematics, The National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511.

Problems

P19.1.1. *Proposed by Lee Peng Yee, National University of Singapore.*

Using only multiplication and the square-root operation on a calculator, we can compute $\sqrt[3]{2}$ as follows. Start with 1, then multiply by 2 and take square-root twice. Repeat the procedure, multiplying the result by 2 and then taking square-root twice. The resulting number tends to $\sqrt[3]{2}$.

Find a similar algorithm for computing $\sqrt[5]{2}$.

The following problems are questions in the International Mathematical Olympiad held in Sigtuna, Sweden, July 1991.

P19.2.1.

Given a triangle ABC, let I be the centre of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

P19.2.2.

Let $n > 6$ be an integer and $a_1, a_2, ..., a_k$ be all the natural numbers less than n and relatively prime to n. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that n must be either a prime or a power of 2.

P19.2.3.

Let $S = \{1, 2, 3, ..., 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.

P19.2.4.

Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, 3, ..., k$ in such a way that at each vertex which belongs to two or more edges the greatest common divisor of the integers labelling those edges is equal to 1.

[A graph G consists of a set of points, called *vertices*, together with a set of edges joining certain pairs of distinct vertices. Each pair of vertices u, v belongs to at most one edge. The graph G is connected if for each pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, v_2, ..., v_m = y$ such that each pair v_i, v_{i+1} $(0 \leq i < m)$ is joined by an edge of G.]

P19.2.5.

Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

P19.2.6.

An infinite sequence $x_0, x_1, x_2, ...$ of real numbers is said to be *bounded* if there is a constant C such that $|x_i| \leq C$ for every $i \geq 0$.

Given any real number $\alpha > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, ...$ such that

$$|x_i - x_j|\,|i - j|^\alpha \geq 1$$

for every pair of distinct non-negative integers i, j.

Solutions

P19.1.1. *Solution by Chiang Kuo Chiang, Former Student, Victoria Junior College*

We first observe that

$$S_n = \frac{1}{n}(1 + S_{n-2} + S_{n-3} + \cdots + S_1),$$
$$S_{n-1} = \frac{1}{n-1}(1 + S_{n-3} + S_{n-4} + \cdots + S_1).$$

Therefore,

$$nS_n - (n-1)S_{n-1} = S_{n-2}.$$

Thus,

$$\begin{aligned}
S_n - S_{n-1} &= (-\frac{1}{n})(S_{n-1} - S_{n-2}) \\
&= (-\frac{1}{n})(-\frac{1}{n-1})(S_{n-2} - S_{n-3}) \\
&= \cdots \\
&= (-\frac{1}{n})(-\frac{1}{n-1})(-\frac{1}{n-2})\cdots(S_2 - S_1) \\
&= (-\frac{1}{n})(-\frac{1}{n-1})(-\frac{1}{n-2})\cdots\left(\frac{1}{2} - 1\right) \\
&= (-\frac{1}{n})(-\frac{1}{n-1})(-\frac{1}{n-2})\cdots\left(-\frac{1}{2}\right) \\
&= \frac{(-1)^{n-1}}{n!}.
\end{aligned}$$

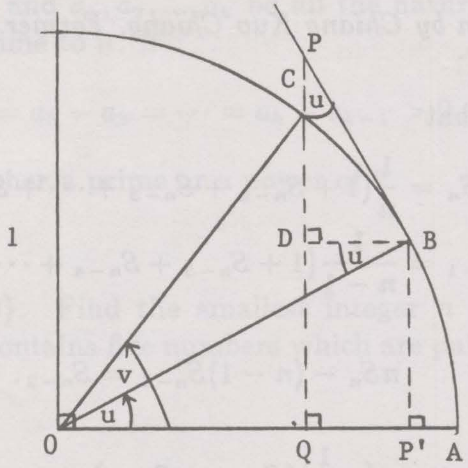
Applying the method of differences, we get

$$S_n - S_1 = -\frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{n-1}}{n!};$$
$$S_n = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{n-1}}{n!}.$$

In the limit when n approaches infinity, we have

$$\begin{aligned}
\lim_{n\to\infty} S_n &= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots \\
&= 1 - e^{-1}.
\end{aligned}$$

P19.1.2. *Solution by Tok Teck Bok, student, Victoria Junior College.*



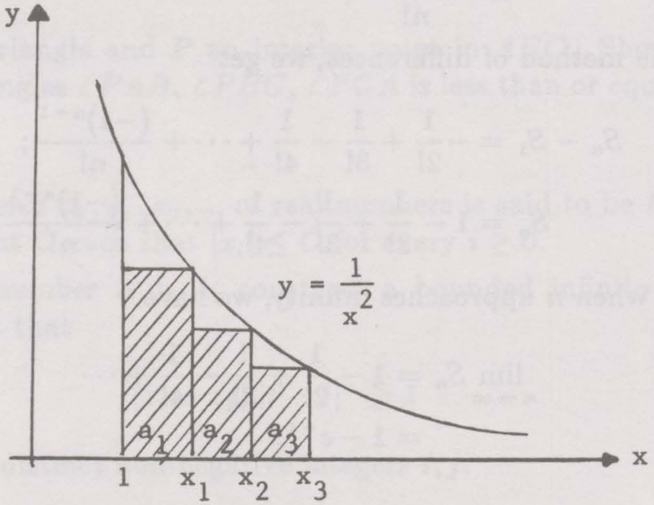
Consider a quadrant of the unit circle. Let P be a point on line segment QC produced such that PB is a tangent to the quadrant.

$$PB \geq arcBC,$$
$$= v - u.$$
$$DB = PB \sin u \geq (v - u)\sin u.$$

But we note that $DB = QP' = \cos u - \cos v$.

Therefore, the result follows.

P19.1.3. *Solution by Tok Teck Bok, student, Victoria Junior College.*



Consider the graph of $y = \frac{1}{x^2}$, $x > 0$.

Let a_i be the width of the i^{th} rectangle under the curve. Then we have

$$\begin{aligned} x_n &= 1 + a_1 + a_2 + a_3 + \cdots + a_n \\ &= 1 + s_n . \end{aligned}$$

$$\begin{aligned} \text{Area of } i^{\text{th}} \text{ rectangle} &= a_i \left(\frac{1}{x_i^2}\right) \\ &= \frac{a_i}{(1+s_i)^2}. \end{aligned}$$

$$\begin{aligned} \text{Thus, the total area of the rectangles} &= \sum_{k=1}^{\infty} \frac{a_k}{(1+s_k)^2} \\ &< \int_1^{\infty} \frac{1}{x^2} dx \\ &= \left[-\frac{1}{x}\right]_1^{\infty} \\ &= 1. \end{aligned}$$

Since $\frac{a_k}{(1+s_k)^2} \to 0$, the result follows by the integral test.

P19.1.4. *Official Solution.*

Join D to A, B and M. Since $\angle CEF = \angle DEG = \angle EMD$ and $\angle ECF = \angle MAD$, $\Delta CEF \sim \Delta AMD$, so $CE \cdot MD = AM \cdot EF$. On the other hand, since $\angle ECG = \angle MBD$ and $\angle CGE = \angle CEF - \angle GCE = \angle EMD - \angle MBD = \angle BDM$, $\Delta CGE \sim \Delta BDM$, so $GE \cdot MB = CE \cdot MD$. Hence we have $GE \cdot MB = AM \cdot EF$, i.e.

$$\frac{GE}{EF} = \frac{AM}{MB} = \frac{tAB}{(1-t)AB} = \frac{t}{1-t}.$$

Remark. If the point M lies between A and E, as shown in the Figure 2, we may interchange the roles of A and B, F and G, and let $t' = 1 - t$. By the same reason, we have

$$\frac{EF}{GE} = \frac{MB}{AM} = \frac{t'}{1-t'} = \frac{1-t}{t},$$

and

$$\frac{GE}{EF} = \frac{t}{1-t}.$$

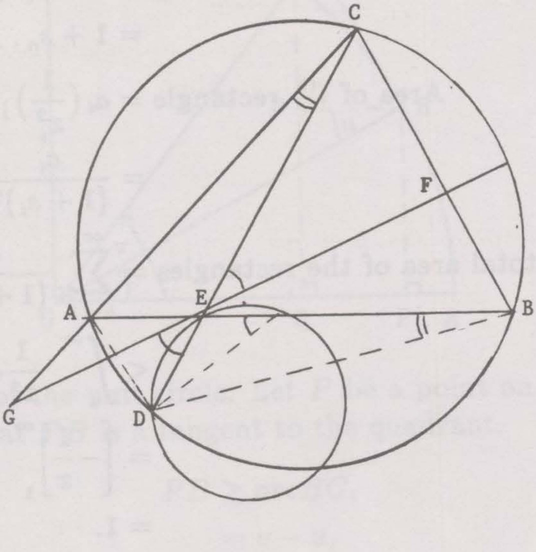


Fig 1.

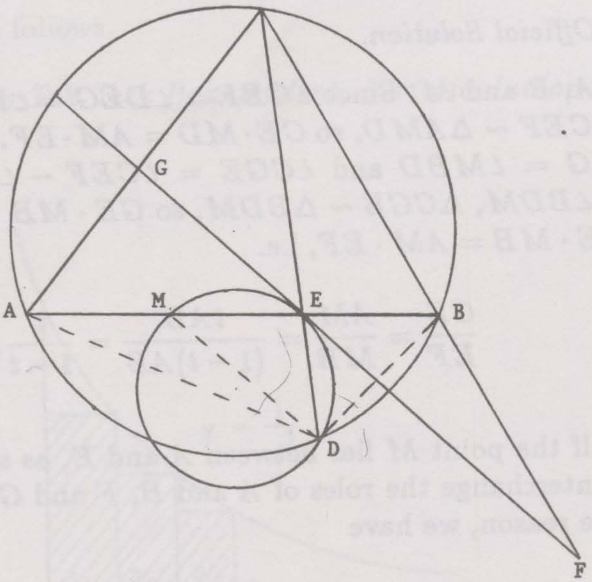


Fig 2.

P19.1.5. *Official Solution.*

According to a certain orientation we denote the given $2n-1$ points in turn by $0, 1, 2, ..., 2n-2$. Let $S = \{0, 1, 2, ..., 2n-2\}$. For $\ell \in Z$, we identify ℓ with $i \in S$ when $\ell \equiv i \pmod{2n-1}$. i and $j(\in S)$ are said to be associated if $|i-j| = n+1$ or $n-2$. Obviously, i and j are associated iff the interior of one of the corresponding arcs contains exactly n of the given $2n-1$ points. If k is a number with the property required in our problem, we simply say that k has $*-$property. Then, k has $*-$property iff for any $A \subset S$ with k elements, there exists two associated elements in A.

(1) n has $*-$property. Let $A \subset S$ and $\#A = n$. Suppose that there is no pair of elements in A which are associated. For $j \in A$, let $S(j)$ be the set of all elements (in S) which are associated with j. Hence $S(j)$ and A are disjoint for each $j \in A$. Since each element occurs at most in two $S(j)$'s, the union of $S(j)$ (where $j \in A$) has at least n elements which contradicts $\#A = n$.

(2) If $3 \not| \, 2n-1, n-1$ has not $*-$property. It sufficies to construct a subset B of S such that $\#B = n-1$ and there is no pair of associated elements in B. Let B and C be

$$\{3\ell \mid \ell = 0, 1, 2, ..., n-2\}, \quad \{3\ell + n - 2 \mid \ell = 0, 1, 2, ..., n-1\}$$

respectively. Since

$$\begin{aligned} C &= \{3(n-1-\ell) + n - 2 \mid \ell = 0, 1, 2, ..., n-1\} \\ &= \{4n - 5 - 3\ell \mid \ell = 0, 1, 2, ..., n-1\} \\ &= \{3(-1-\ell) \mid \ell = 0, 1, 2, ..., n-1\} \end{aligned}$$

$3 \not| \, 2n-1$ implies that $B \cup C = \{3\ell \mid \ell = -n, -n+1, ..., n-2\} = S$. Hence $\#B = n-1$, $\#C = n$, B and C are disjoint. Noticing that for $j \in B$, $j = 3\ell$ where $0 \le \ell \le n-2$ and $S(j) = \{3\ell + n - 2, 3\ell + n + 1\} = \{3\ell + n - 2, 3(\ell+1) + n - 2\} \subset C$, we conclude that $S(j)$ and B are disjoint. This means that there is no pair of elements in B which are associated.

(3) Now we consider the case $3 | 2n-1$. In this case $n \equiv 2 \pmod 3$. Let $n = 3m-1$ where $m \ge 2$. Then $2n-1 = 3(2m-1)$, $n-2 = 3(m-1)$ and $n+1 = 3m$. If i and $j(\in S)$ are associated, i and j are congruent modulo 3. We write S_r for $\{3\ell + r \mid \ell = 0, 1, 2, ..., 2m-2\}$ $(r = 0, 1, 2)$. Thus S is partitioned into three disjoint subsets S_0, S_1, S_2. For each i in S, $i \in S_r$

implies that two elements associated with i are also contained in S_r. We claim that $3m-2$ has $*$−property. Indeed, if $A \subset S$ and $\#A = 3m-2$, by the pigeonhole principle one subset of the form $A \cap S_r$ has at least m members. An argument similar to that used in (1) allows us to deduce that there is a pair of associated elements in $A \cap S_r \subset A$.

We show that $3(m-1)$ has not $*$−property. It suffices to construct an $(m-1)$–element subset of S_r which has no pair of associated elements. Let $B_r = \{3\ell + r \mid \ell = 0, 1, ..., m-2\}$ $(r = 0, 1, 2)$. Noticing that i and j are associated iff $|i-j| = 3(m-1)$ or $3m$, we see that there is no pair of associated elements in each of B_0, B_1, B_2.

Summing up, we conclude that

$$\min\{k|k \text{ has } * -\text{property}\} = \begin{cases} n & \text{if } 3 \not| \, 2n-1 \\ n-1 & \text{if } 3|2n-1. \end{cases}$$

P19.1.6. *Official Solution.*

Since 2^n+1 is odd, $n^2|2^n+1$ implies that n is odd. Let n be an integer with the property that $n^2|2^n+1$ and $n \geq 3$. Now let p be the smallest prime factor of n. Then $p \geq 3$ and $p|2^n+1$, i.e. $2^n \equiv -1 \pmod p$. Let i be the smallest natural number such that $2^i \equiv -1 \pmod p$. It follows from $2^{p-1} \equiv 1 \pmod p$ that $i < p-1$. Let $n = ki + r$ where $0 \leq r \leq i-1$. Then we have $2^n = 2^{ki} \times 2^r \equiv (-1)^k 2^r \pmod p$. Suppose that k is even. $2^r \equiv 2^n \equiv -1 \pmod p$. It follows that $r = 0$ and $2 \equiv 0 \pmod p$ which contradicts that $p \geq 3$. Thus k is odd and $2^r \equiv 1 \pmod p$. We claim that $r = 0$. Suppose that $r > 0$. Let $i = r + d$ where $1 \leq d < i$. $2^d \equiv 2^r \times 2^d \equiv 2^i \equiv -1 \pmod p$ which contradicts the choice of i. Hence $n = ki$ and $i|n$. Since $i < p$ and p is the smallest prime factor, $i = 1$. It follows that $p|2+1$. We have that $p = 3$.

Let $n = 3^k d$ where $k \geq 1$ and $(d, 3) = 1$. We shall prove that $k = 1$. Suppose that $k \geq 2$. Since $n^2|2^n+1$, $3^{2k}|(3-1)^n+1$. This implies that

$$3^{k+2}|(3^{k+1}d - \sum_{h=2}^{k+1}(-1)^h C_n^h 3^h).$$

Noticing that the exponent of 3 in the decomposition of $h!$ is less than $\frac{h}{2}$, we see that the exponent of 3 in the decomposition $3^h C_n^h$ is greater than $k + \frac{h}{2}$, hence $3^{k+2}|3^h C_n^h$ for $h \geq 2$. We deduce that $3^{k+2}|3^{k+1}d$ which is a contradiction. Thus $n = 3d$.

We claim that $n = 3$. Suppose that $d \neq 1$. Let q be the smallest prime factor of d. Since d is odd and $(d, 3) = 1$, $q \geq 5$, $q | 2^n + 1$. Let j be the smallest natural number such that $2^j \equiv -1 \pmod{q}$. By Fermat's theorem $j < q - 1$. An agrument similar to that in the first paragraph gives $j | n$. Since $n = 3d$, $(3, d) = 1$, $j < q - 1$ where q is the smallest prime factor of d, hence $j = 1$ or 3. The congruence $2^j \equiv -1 \pmod{q}$ yields that $q | 3$ or $q | 9$. Hence $q = 3$. This contradicts the fact that $q \geq 5$.

Summarizing, we have proved that the only possible case of $n^2 | 2^n + 1$ for $n \geq 3$ is $n = 3$.

Finally, it is easy to verify that $n^2 | 2^n + 1$ for $n = 1$ or 3.

P19.1.7. *Official Solution.*

If $f(y_1) = f(y_2)$, the functional equation implies that $y_1 = y_2$. Letting $y = 1$ gives $f(1) = 1$. Letting $x = 1$ gives $f(f(y)) = \frac{1}{y}$ for all $y \in Q^+$. Applying f to this implies that $f(\frac{1}{y}) = \frac{1}{f(y)}$ for all $y \in Q^+$. Finally setting $y = f(\frac{1}{t})$ yields $f(xt) = f(x)f(t)$ for all $x, t \in Q^+$.

Conversely it is easy to see that any f satisfying (a) $f(xt) = f(x)f(t)$ (b) $f(f(x)) = \frac{1}{x}$ for all $x, t \in Q^+$ solves the functional equation.

A function $f : Q^+ \to Q^+$ satisfying (a) can be constructed by defining arbitrarily on the prime numbers and extending as

$$f(p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}) = (f(p_1))^{n_1} (f(p_2))^{n_2} \ldots (f(p_k))^{n_k}$$

where p_j denotes the j–th prime number and $n_j \in Z$. Such a function will satisfy (b) if and only if it satisfies (b) for each prime number.

A possible construction is as follows:

$$f(p_j) = \begin{cases} p_{j+1} & \text{if } j \text{ is odd} \\ \dfrac{1}{p_{j-1}} & \text{if } j \text{ is even .} \end{cases}$$

Extending it as above, we get a function $f : Q^+ \to Q^+$. Clearly $f(f(p)) = \frac{1}{p}$ for each prime p, hence f satisfies the functional equation.

P19.1.8. *Official Solution.*

Denote by W the set of all the natural number n_0 such that beginning with n_0 the player A can manage to win the game. We have the following

Lemma. Suppose $\{m, m+1, ..., 1990\} \subset W$, $s \leq 1990$ and $\frac{s}{p^r} \geq m$, where p^r is the largest factor of s such that p is a prime and $r \in \mathbb{N}$. Then all the natural numbers n_0 such that

$$\sqrt{s} \leq n_0 < m$$

are in W.

Proof. Beginning with such an n_0, player A may take $n_1 = s$. Then the player B has to choose a natural number n_2 such that

$$m \leq \frac{s}{p^r} \leq n_2 < s \leq 1990.$$

Because $n_2 \in W$, certainly the player A can win the game afterwards. The lemma is proved.

Because $45^2 = 2025 > 1990$, it is evident that all n_0's such that $45 \leq n_0 \leq 1990$ are in W. Since $m = 45$ and $s = 420 = 2^2 \times 3 \times 5 \times 7$ satisfy the hypothesis of the above lemma and

$$\sqrt{420} < 21 \leq 45,$$

it follows that

$$\{21, 22, ..., 44\} \subset W.$$

Using the lemma with $m = 21$ and $s = 168 = 2^3 \times 3 \times 7$, we see that

$$\{13, 14, ..., 20\} \subset W.$$

For $m = 13$ and $s = 105 = 3 \times 5 \times 7$, it follows from the lemma that

$$\{11, 12\} \subset W.$$

By letting $m = 11$ and $s = 60 = 2^2 \times 3 \times 5$, the following can be shown:

$$\{8, 9, 10\} \subset W.$$

We have proved that

$$\{8, 9, ..., 1990\} \subset W.$$

For $n_0 > 1990$ player A can choose a natural number r such that

$$2^r \times 3^2 < n_0 \leq 2^{r+1} \times 3^3 < n_0^2$$

and then take

$$n_1 = 2^{r+1} \times 3^2.$$

Now, no matter what the choice of player B is, it certainly follows that

$$8 \leq n_2 < n_0.$$

After a finite number of steps, the situation will be

$$8 \leq n_{2k} \leq 1990$$

and the player A will win the game. We conclude that for $n_0 \geq 8$ the player A can manage to win the game.

Now consider the case $n_0 \leq 5$. Because the smallest product of three different prime numbers is $2 \times 3 \times 5 = 30 > 5^2$, the player A has to take

$$n_1 = p^r \times q^s,$$

where p is a prime, q is a prime or 1, $p^r > q^s$ and $r, s \geq 1$. Then player B can choose

$$n_2 = q^2 = \frac{n_1}{p^r} < \sqrt{n_1} \leq n_0.$$

After *a* finite number of steps, player B will get $n_{2k} = 1$ and win the game.

For $n_0 = 6$ or $n_0 = 7$, player A has to choose $n_1 = 30 = 2 \times 3 \times 5$ or $n_1 = 42 = 2 \times 3 \times 7$ and then player B has to choose $n_2 = 6$. Afterwards, A and B have to choose

$$30, 6, 30, 6, 30, 6, \ldots$$

alternatively to avoid losing the game and they end with a tie.

P19.1.9. ***Official Solution.***

Suppose 1990-gon $A_0 A_1 A_2 \ldots A_{1989}$ has the properties (i) and (ii). We express vector $\overrightarrow{A_r A_{r+1}}$ as a complex number

$$n_r e^{ir\alpha}$$

where $\alpha = \dfrac{2\pi}{1990}$, $r \in \{0, 1, \ldots, 1989\}$ and A_{1990} denotes A_0. Then $n_0, n_1, \ldots, n_{1989}$ should be a permutation of numbers $1^2, 2^2, \ldots, 1990^2$. The problem can be reformulated as follows: find a permutation $(n_0, n_1, ..., n_{1989})$ of the numbers $1^2, 2^2, ..., 1990^2$, such that

$$\sum_{r=0}^{1989} n_r e^{ir\alpha} = 0.$$

For convenience, we call n_r's "weights". Suppose a unit disk lying horizontally is supported by a pinpoint at origin 0. The problem is : how can we put the weights $1^2, 2^2, ..., 1990^2$ on the boundary of the disk at the points which divide the circumference into equal arcs to get a balanced system?

To begin with, we divide the 1990 weights into 995 pairs

$$(1^2, 2^2), (3^2, 4^2), ..., (1989^2, 1990^2),$$

and put each pair at two endpoints of certain diameter. The problem is now reduced to a simpler one: how can we put 995 weights

$$2^2 - 1^2 = 3,\ 4^2 - 3^2 = 7,\ 6^2 - 5^2 = 11, ...$$

$$..., 1990^2 - 1989^2 = 3979$$

at the points dividing the unit circumference into 995 equal arcs to get a balanced system? Noticing

$$995 = 5 \times 199,$$

we divide the 995 weights into 199 groups

$$(*) \begin{cases} (3, 7, 11, 15, 19), (23, 27, 31, 35, 39), \\ \dots, (3963, 3967, 3971, 3975, 3979). \end{cases}$$

Let $\beta = \frac{2\pi}{199}$, $\gamma = \frac{2\pi}{5}$. We denote by F_1 the pentagon with vertices $1, e^{i\gamma}, e^{2i\gamma}, e^{3i\gamma}, e^{4i\gamma}$, and by F_{k+1} the pentagon $e^{ik\beta} F_1$. Putting the five weights of the $(k+1)$st group in (*) at the vertices of the pentagon F_{k+1}, we obtain the $(k+1)$st group of complex numbers:

$$(2k+3)e^{ik\beta}, (2k+7)e^{i(k\beta+\gamma)}, (2k+11)e^{i(k\beta+2\gamma)}$$

$$(2k+15)e^{i(k\beta+3\gamma)}, (2k+19)e^{i(k\beta+4\gamma)},$$

where $k = 0, 1, 2, ..., 198$. Noticing that

$$1 + e^{i\gamma} + e^{2i\gamma} + e^{3i\gamma} + e^{4i\gamma} = 0,$$

we can write the sum of the complex numbers of the $(k+1)$st group as

$$\eta e^{ik\beta},$$

where

$$\eta = 3 + 7e^{i\gamma} + 11e^{2i\gamma} + 15e^{3i\gamma} + 19e^{4i\gamma}.$$

The total sum of all the 199 groups of complex numbers is

$$\eta(1 + e^{i\beta} + ... + e^{198i\beta}) = 0.$$

We come to the conclusion that there certainly exists a 1990-gon with the required properties.

Finally, we pointed out that the solution can be straightened out, neatly, as follows

$$\begin{aligned} 0 &= \sum_{k=0}^{198} \sum_{\ell=0}^{4} (20k + 4\ell + 3) e^{i(k\beta + \ell\gamma)} \\ &= \sum_{k=0}^{198} \sum_{\ell=0}^{4} [(10k + 2\ell + 2)^2 - (10k + 2\ell + 1)^2] e^{i(k\beta + \ell\gamma)} \\ &= \sum_{k=0}^{198} \sum_{\ell=0}^{4} \sum_{m=1}^{2} (10k + 2\ell + m)^2 e^{i(k\beta + \ell\gamma + m\pi)} \end{aligned}$$

When k passes through $0, 1, ..., 198$, ℓ through $0, 1, 2, 3, 4$, and m through $1, 2$, the expression

$$10k + 2\ell + m$$

passes through $1, 2, ..., 1990$, taking each value exactly once, and the expression

$$e^{i(k\beta + \ell\gamma + m\pi)} = e^{i \frac{10k + 398\ell + 995m}{1990} 2\pi}$$

takes $1, e^{i\alpha}, ..., e^{1989i\alpha}$ each once.